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PROBABILITY INEQUALITIES FOR SUMS OF INDEPENDENT RANDOM VARIABLES WITH VALUES IN A BANACH SPACE*

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In the present paper we improve in a certain sense the inequalities obtained in [1]. Partially, these inequalities have been communicated in [2] (see below Theorem 1 and Corollary 1). A progress in comparison with [1] is achieved by the modification of the method of proof.

Thus, let $X_1, X_2, \ldots, X_n$ be independent random variables with values in a separable Banach space B with norm $|\cdot|$. We set $S_k=\sum_1^k X_i$, $M_n=\max_{1\leqslant k\leqslant n}|S_k|$. Let α be any number such that $P(2M_n\geqslant\alpha)<1$. For the sake of brevity we set

$$\delta=\mathbf{P}(2M_n\geqslant\alpha),\quad k_\alpha=[y/\alpha],$$
$$P(y)=\min\left[\delta,\ \sum_1^n \mathbf{P}(|X_i|\geqslant y)\right].$$

THEOREM 1. For any $y\geqslant\alpha$ and $1\geqslant\delta_1\geqslant\delta$ we have

$$\mathbf{P}(M_n\geqslant y)\leqslant\sum_1^{k_\alpha-1} kP\left(\frac{k_\alpha-k}{k}\alpha\right)\delta^{k-1}+\delta^{k_\alpha}\leqslant\delta_1^{-2}\int_0^{y/\alpha-1}u\delta_1^uP\left(\frac{y-(u+1)\alpha}{u}\right)du+\delta_1^{y/\alpha-2}y/\alpha,\tag{1}$$

where $\sum_1^0=0$.

We introduce the notations $A_t=\sum_1^n\mathbf{E}|X_j|^t$, $A_g=\sum_1^n\mathbf{E}\exp\{g(|X_j|)\}$, where g is an arbitrary Borel function.

COROLLARY 1. For any $t>0$, $1\geqslant\gamma\geqslant\delta$, $y\geqslant\alpha$ we have

$$\mathbf{P}(M_n\geqslant y)\leqslant c(t,\gamma)A_t/y^t+2(y/\alpha)^2\gamma^{y/2\alpha-2},\tag{2}$$

where $c(t,\gamma)=2^t\Gamma(t+2)\gamma^{-2}(-\ln\gamma)^{-t-2}$.

Proof. Without loss of generality, we can assume that $y\geqslant 2\alpha$. First we estimate the integral

$$I\equiv\int_0^{y/\alpha-1}u\delta^uP\left(\frac{y-(u+1)\alpha}{u}\right)du=\int_0^{y/2\alpha-1}+\int_{y/2\alpha-1}^{y/\alpha-1}\equiv I_1+I_2.\tag{3}$$

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If $0 \leqslant u \leqslant y/2\alpha - 1$, then $y-(u+1)\alpha \geqslant y/2$. Consequently, for these values of u we have

$$P\left(\frac{y-(u+1)\alpha}{u}\right) \leqslant A_t (2u/y)^t.$$

As a result we obtain the estimate

$$I_1 \leqslant A_t (2/y)^t \int_0^\infty u^{t+1} e^{u \ln \gamma} du = A_t (2/y)^t (-\ln \gamma)^{-t-2} \Gamma(t+2).$$

On the other hand,

$$I_2 \leqslant \gamma \int_{y/2\alpha-1}^{y/\alpha-1} u \gamma^u du < (y/\alpha)^2 \gamma^{y/2\alpha}. \tag{4}$$

Thus,

$$I \leqslant c(t,\gamma) A_t / y^t + (y/\alpha)^2 \gamma^{y/2\alpha}.$$

From this estimate, by virtue of (1), there follows inequality (2).

COROLLARY 2. Assume that the continuous function $g(y)\uparrow$ and $g(0) = 0$. Then

$$(\forall (y \geqslant \alpha,\ 1 \geqslant \gamma \geqslant \delta))\ (\mathbf{P}(M_n \geqslant y) \leqslant P_0(y,\alpha,\gamma) + 2(y/\alpha)^2 \gamma^{y/2\alpha-2}), \tag{5}$$

where

$$P_0(y,\alpha,\gamma) = \begin{cases} A_g \gamma^{-2} (y^2/8\alpha^2) \exp\{-g(y/2u_0)\}, \\ \quad \text{if} \quad g(\alpha) \leqslant -(y/2\alpha)\ln\gamma, \\ A_g e^{-g(\alpha)} \gamma^{-2} (-\ln\gamma)^{-2}, \\ \quad \text{if} \quad g(\alpha) \geqslant -(y/2\alpha)\ln\gamma, \end{cases}$$

u_0 being the root of the equation $g(y/2u) + u\ln\gamma = 0$.

Proof. Let I, I_1, and I_2 be defined in the same way as in the proof of Corollary 1.

Since $y-(u+1)\alpha \geqslant y/2$ for $0 \leqslant u \leqslant y/2\alpha - 1$, we have

$$P\left(\frac{y-(u+1)\alpha}{u}\right) \leqslant A_g e^{-g(y/2u)}.$$

Therefore,

$$I_1 \leqslant A_g \int_0^{y/2\alpha} u \exp\{-g(y/2u) + u\ln\gamma\}\, du.$$

It is easy to see that

$$g(y/2u_0) - u_0 \ln\gamma \leqslant 2(g(y/2u) - u\ln\gamma).$$

Thus,

$$I_1 \leqslant A_g (y^2/8\alpha^2) \exp\{-g(y/2u_0)\}.$$

If $g(\alpha) \geqslant -(y/2\alpha)\ln\gamma$, then $g(\alpha) \geqslant g(y/2u_0)$ and

$$I_1 \leqslant A_g e^{-g(\alpha)} \int_0^{y/2\alpha} u e^{u \ln\gamma} du < A_g e^{-g(\alpha)} (-\ln\gamma)^{-2}.$$

As far as I_2 is concerned, we make use of the available estimate (4). Returning now to the formulas (1), (3), we obtain inequality (5).

We proceed to the proof of Theorem 1. For this we need several auxiliary statements. Let $\tau_1 = \min\{k\colon 1 \leqslant k \leqslant n,\ |S_k| \geqslant \alpha\}$ if $\{k\colon 1 \leqslant k \leqslant n,\ |S_k| \geqslant \alpha\} \neq \varnothing$. Otherwise we set $\tau_1 = n+1$. Further we define τ_j for $j > 1$ by induction, setting

$$\tau_j = \min\{k\colon |S_k| \geqslant |S_{\tau_{j-1}}| + \alpha,\ 1 \leqslant k \leqslant n\},$$

if the corresponding set of values of k is nonempty. Otherwise, $\tau_j = n+1$.

Obviously, $\tau_j \leqslant \tau_{j+1}$ and $|S_{\tau_k}| \geqslant k\alpha$, if $\tau_k \leqslant n$.

We set

$$Y_1 = |S_{\tau_1}| - \alpha,$$
$$Y_j = |S_{\tau_j}| - |S_{\tau_{j-1}}| - \alpha, \quad j > 1,$$

if $\tau_j \leqslant n$. For $\tau_j = n + 1$ we have $Y_j = 0$.

Let $\mathscr{F}_k$ be the σ-algebra generated by the random variables $X_1, X_2, \ldots, X_k$ and let $\mathscr{B}_k$ be the σ-algebra generated by the Markov moment τ_k.

LEMMA 1. For any $j \leqslant m \leqslant n$ we have

$$\mathbf{P}(\tau_m \leqslant n/\mathscr{B}_j) \leqslant \delta^{m-j}\mathbf{P}(\tau_j \leqslant n/\mathscr{B}_j).$$

Proof. By definition,

$$\{\tau_{k-1} = i, \quad \tau_k \leqslant n\} = \left\{\tau_{k-1} = i, \quad \max_{i<l\leqslant n} |S_l| \geqslant |S_i| + \alpha\right\}.$$

On the other hand,

$$|S_l| \geqslant |S_i| + \alpha \Rightarrow \left|\sum_{i+1}^{l} X_q\right| \geqslant \alpha.$$

Setting $D_k = \{\tau_k \leqslant n\}$, $D_{k,i} = \{\tau_k = i\}$, $E_i = \left\{\max_{i<l\leqslant n}\left|\sum_{i+1}^{l} X_q\right| \geqslant \alpha\right\}$, we have the inclusion

$$D_{k-1,\,i}D_k \subset D_{k-1}E_i. \tag{6}$$

Let $\Lambda \in \mathscr{B}_j$, $j < k - 1$. Then

$$\mathbf{P}(D_{k-1,i}E_i\Lambda) = \sum_{l=1}^{i-1} \mathbf{P}(D_{k-1,i}E_iD_{j,l}\Lambda). \tag{7}$$

By definition $D_{i,\,l}\Lambda \in \mathscr{F}_l$. Therefore,

$$\mathbf{P}(D_{k-1,\,i}E_iD_{j,\,l}\Lambda) = \mathbf{P}(E_i)\mathbf{P}(D_{k-1,\,i}D_{j,\,l}\Lambda). \tag{8}$$

From the equalities (7), (8) there follows that

$$\mathbf{P}(D_{k-1,\,i}E_i\Lambda) = \mathbf{P}(E_i)\mathbf{P}(D_{k-1,\,i}\Lambda).$$

This means that

$$\mathbf{P}(D_{k-1,\,i}E_i/\mathscr{B}_j) = \mathbf{P}(E_i)\mathbf{P}(D_{k-1,\,i}/\mathscr{B}_j). \tag{9}$$

Obviously,

$$\{\tau_k \leqslant n\} = D_k \bigcup_{i=1}^{n-1} D_{k-1,i}. \tag{10}$$

By virtue of the formulas (6), (9), (10) we have

$$\mathbf{P}(\tau_k \leqslant n/\mathscr{B}_j) = \sum_{i=1}^{n-1} \mathbf{P}(D_{k-1,i}D_k/\mathscr{B}_j) \leqslant \sum_{i=1}^{n-1} \mathbf{P}(D_{k-1,i}E_i/\mathscr{B}_j) = \sum_{i=1}^{n-1} \mathbf{P}(E_i)\mathbf{P}(\tau_{k-1} = i/\mathscr{B}_j). \tag{11}$$

It is easy to see that for any k we have

$$\max_{k<l\leqslant n}\left|\sum_{k+1}^{l} X_i\right| \leqslant 2M_n.$$

Therefore,

$$\mathbf{P}(E_k) \leqslant \mathbf{P}(2M_n \geqslant \alpha) = \delta. \tag{12}$$

From (11), by virtue of the relation (12), there follows that

$$\mathbf{P}(\tau_k \leqslant n/\mathscr{B}_j) \leqslant \delta\mathbf{P}(\tau_{k-1} \leqslant n - 1/\mathscr{B}_j) \leqslant \delta\mathbf{P}(\tau_{k-1} \leqslant n/\mathscr{B}_j), \quad j < k - 1. \tag{13}$$

If $\Lambda \in \mathscr{B}_j$, then $\mathbf{P}(D_{j,\,i}E_i\Lambda) = \mathbf{P}(E_i)\mathbf{P}(D_{j,\,i}\Lambda)$. Thus, equality (9) is satisfied also for k − 1 = j. Consequently, inequalities (13) remain valid also for k − 1 = j.

Applying successively inequalities (13) for $= \overline{m, j+1}$, we obtain the assertion of the lemma.

COROLLARY 3. For any $k \leqslant n$ we have

$$\mathbf{P}(\tau_k \leqslant n) \leqslant \delta^k.$$

LEMMA 2. For any $y > 0$ we have

$$\mathbf{P}(Y_k \geqslant y) \leqslant \delta^{k-1}\mathbf{P}(y).$$

Proof. Obviously,

$$\mathbf{P}(Y_k \geqslant y) = \sum_{i=1}^{n} \mathbf{P}(Y_k \geqslant y,\ \tau_k = i).$$

Further,

$$\mathbf{P}(Y_k \geqslant y,\ \tau_k = i) \leqslant \mathbf{P}(|S_i| - |S_{i-1}| \geqslant y,\ \tau_{k-1} < i) \leqslant \mathbf{P}(|X_i| \geqslant y,\ \tau_{k-1} < i) = \mathbf{P}(|X_i| \geqslant y)\mathbf{P}(\tau_{k-1} < i).$$

Thus,

$$\mathbf{P}(Y_k \geqslant y) \leqslant \mathbf{P}(\tau_{k-1} \leqslant n) \sum_{1}^{n} \mathbf{P}(|X_i| \geqslant y).$$

Applying now Corollary 3, we obtain that

$$\mathbf{P}(Y_k \geqslant y) \leqslant \delta^{k-1} \sum_{1}^{n} \mathbf{P}(|X_i| \geqslant y). \tag{14}$$

On the other hand, by virtue of the same Corollary 3,

$$\mathbf{P}(Y_k \geqslant y) \leqslant \mathbf{P}(\tau_k \leqslant n) \leqslant \delta^k. \tag{15}$$

From the inequalities (14) and (15) there follows the assertion of the lemma.

Proof of Theorem 1. First we assume that $y = m\alpha$, where m is an integer.

Let

$$A_k = \{\tau_k \leqslant n,\ |S_{\tau_{k-1}}| < m\alpha \leqslant |S_{\tau_k}|\},\quad k \geqslant 1.$$

Here $\tau_0 = 0$, $S_0 = 0$.

It is easy to see that

$$\{M_n \geqslant m\alpha\} = \bigcup_{1}^{m} A_k. \tag{16}$$

Further,

$$A_k \subset \left\{\tau_k \leqslant n,\ \sum_{1}^{k} Y_j \geqslant (m-k)\alpha\right\}. \tag{17}$$

From here

$$(\forall\,(0 < k < m))\ \mathbf{P}(A_k) \leqslant \sum_{1}^{k} \mathbf{P}(Y_j \geqslant (m-k)\alpha/k,\ \tau_k \leqslant n). \tag{18}$$

Making now use of Lemmas 1 and 2, we have

$$(\forall\,(0 < k < m))\ (\mathbf{P}(Y_j \geqslant (m-k)\alpha/k,\ \tau_k \leqslant n) \leqslant \delta^{k-j}\mathbf{P}(Y_j \geqslant \alpha(m-k)/k,\ \tau_j \leqslant n) \leqslant \delta^{k-1}\mathbf{P}((m-k)\alpha/k)). \tag{19}$$

On the other hand, by virtue of Corollary 3 we have

$$\mathbf{P}\left(\sum_{1}^{m} Y_j \geqslant 0,\ \tau_m \leqslant n\right) = \mathbf{P}(\tau_m \leqslant n) \leqslant \delta^m. \tag{20}$$

From formulas (16)-(20) we obtain

$$\mathbf{P}(M_n \geqslant m\alpha) \leqslant \sum_{1}^{m-1} k\delta^{k-1}\mathbf{P}\left(\frac{(m-k)\,\alpha}{k}\right) + \delta^m. \tag{21}$$

Observing that

$$(\forall\,(y > 0))\,(\mathbf{P}(M_n \geqslant y) \leqslant \mathbf{P}(M_n \geqslant k_\alpha \alpha)),$$

and setting in (21) $m = k_\alpha$, we obtain the first of the inequalities (1).

It is easy to see that

$$\left(\forall\,(1\geqslant\delta_1\geqslant\delta)\right)k\delta^{k-1}P\left(\left(\frac{m}{k}-1\right)\alpha\right)<\delta_1^{-2}\int\limits_k^{k+1}u\delta_1^uP\left(\left(\frac{m}{u}-1\right)\alpha\right)du.$$

From here

$$(\forall\,(y\geqslant 2\alpha))\left(\sum_1^{k_\alpha-2}k\delta^{k-1}P\left(\frac{k_\alpha-k}{k}\alpha\right)<\delta_1^{-2}\int\limits_1^{k_\alpha-1}u\delta_1^uP\left(\left(\frac{k_\alpha}{u}-1\right)\alpha\right)du<\delta_1^{-2}\int\limits_0^{y/\alpha-1}u\delta_1^uP\left(\frac{y-(u+1)\alpha}{u}\right)du\right). \tag{22}$$

From (21), (22) there follows the second of the inequalities (1) for $y\geqslant 2\alpha$. If $\alpha\leqslant y<2\alpha$, then

$$(\forall\,(1\geqslant\delta_1\geqslant\delta))\,(\mathbf{P}\,(M_n\geqslant y)\leqslant\delta\leqslant\delta_1^{y/\alpha-1}y/\alpha).$$

Theorem 1 is completely proved.

LEMMA 3. If the random variables X_j are symmetric, then

$$\mathbf{P}\,(M_n\geqslant y)\leqslant 2\mathbf{P}\,(|S_n|\geqslant y). \tag{23}$$

The proof can be found in [3-5].

We introduce the notations

$$P_1\,(y,\,\alpha,\,\gamma)=\sum_1^{k_\alpha-1}kP\left(\frac{k_\alpha-k}{k}\alpha\right)\gamma^{k-1}+\gamma^{k_\alpha},$$

$$P_2\,(y,\,\alpha,\,\gamma)=\gamma^{-2}\int\limits_0^{y/\alpha-1}u\gamma^uP\left(\frac{y-(u+1)\,\alpha}{u}\right)du+(y/\alpha)\,\gamma^{y/\alpha-2},$$

$$P_3(y,\,\alpha,\,\gamma)=2^t\Gamma(t+2)\gamma^{-2}(-\ln\gamma)^{-t-2}A_t/y^t+2\,(y/\alpha)^2\gamma^{y/2\alpha-2},$$

$$P_4(y,\,\alpha,\,\gamma)=P_0(y,\,\alpha,\,\gamma)+2\,(y/\alpha)^2\gamma^{y/2\alpha-2},$$

$$\beta=\mathbf{P}\,(2|S_n|\geqslant\alpha).$$

COROLLARY 4. Assume that the random variables X_j are symmetric and $\beta<1/2$. Then

$$(\forall\,(y\geqslant\alpha))\,(\mathbf{P}\,(|S_n|\geqslant y)\leqslant P_j(y,\,\alpha,\,2\beta),\quad j=1\div 4). \tag{24}$$

Proof. As a consequence of the inequality (23) we have

$$\delta\leqslant 2\beta. \tag{25}$$

It remains to use Theorem 1 and Corollaries 1, 2.

LEMMA 4. For any integer m > 1 we have

$$\mathbf{P}\,(M_n\geqslant y)\leqslant\mathbf{P}^m\,(M_n\geqslant y/4m)+(\mathbf{P}\,(M_n<y/4m))^{-1}\sum^n\mathbf{P}\,(|X_j|>y/2m)$$

(see [1, Proposition 4]).

COROLLARY 5. For any integer m > 1 and $y\geqslant 4m\alpha$ we have

$$\mathbf{P}\,(M_n\geqslant y)\leqslant P_j^m\,(y/4m,\,\alpha,\,\delta)+(\mathbf{P}(M_n<y/4m))^{-1}\sum_1^n\mathbf{P}(|X_j|>y/2m),\ \ j=1\div 4. \tag{26}$$

Inequality (26) is obtained easily from Theorem 1, Corollaries 1, 2, and Lemma 4.

COROLLARY 6. Assume that the random variables X_j are symmetric and $\beta<1/2$. Then for any integer m and $y\geqslant 4m\alpha$ we have

$$P\,(|S_n|\geqslant y)\leqslant P_j^m\,(y/4m,\,\alpha,\,2\beta)+(1-2\beta)^{-1}\sum_1^n\mathbf{P}\,(|X_j|\geqslant y/2m),\ \ j=1\div 4. \tag{27}$$

Inequality (27) is derived from formulas (25), (26) and the estimate

$$\mathbf{P}\,(M_n\geqslant y/4m)<2\mathbf{P}\,(|S_n|\geqslant\alpha)$$

for $y\geqslant 4m\alpha$, which is a consequence of inequality (23).

For any random variable X, we denote its symmetrization by X^s.

LEMMA 5. Let X be any random variable with values in B. Then

$$(\forall (u > v))\left(\tfrac{1}{2}\mathbf{P}(|X^s| \geqslant 2u) \leqslant \mathbf{P}(|X| \geqslant u) \leqslant \mathbf{P}(|X^s| \geqslant u - v)/P(|X| \leqslant v)\right). \tag{28}$$

The proof of Lemma 5 can be found, for example, in [1, p. 160].

We set $P(y, \gamma) = \min\left[\gamma, \sum_1^n \mathbf{P}(|X_i| \geqslant y)\right]$,

$$P_5(y, \alpha, \gamma) = \sum_1^{[y]-1} 2^k k P\left(\frac{[y]-k}{k}\alpha, \gamma\right)\gamma^{k-1} + (2\gamma)^{[y]}.$$

THEOREM 2. Let $\beta < 1/4$. Then

$$(\forall (y \geqslant 2\alpha))\left(\mathbf{P}(|S_n| \geqslant y) \leqslant \frac{1}{1-\beta} P_5\left(\frac{y-\alpha}{2\alpha}, \alpha, 2\beta\right)\right). \tag{29}$$

Proof. First of all, by virtue of Lemma 5 we have

$$(\forall (y \geqslant \alpha))\left(\mathbf{P}(|S_n| \geqslant y) \leqslant \mathbf{P}(|S_n^s| \geqslant y - \alpha)/\mathbf{P}(|S_n| \leqslant \alpha)\right), \tag{30}$$

$$(\forall (y \geqslant \alpha))\left(\mathbf{P}(|S_n^s| \geqslant \alpha) \leqslant 2\mathbf{P}(2|S_n| \geqslant \alpha) = 2\beta < 1/2\right). \tag{31}$$

Inequality (31) allows us to apply Corollary 4 to $\mathbf{P}(|S_n^s| \geqslant y - \alpha)$; as a result we obtain

$$(\forall (y \geqslant 3\alpha))\left(\mathbf{P}|S_n^s| \geqslant y - \alpha) \leqslant P_{1s}(y - \alpha, 2\alpha, 2\beta_s)\right). \tag{32}$$

Here

$$P_{1s}(y, \alpha, \gamma) = \sum_1^{k_\alpha - 1} k P_s\left(\frac{k_\alpha - k}{k}\alpha, \gamma\right)\gamma^{k-1} + \gamma^{k_\alpha}, \quad k_\alpha = [y/\alpha], \quad y \geqslant \alpha.$$
$$P_s(y, \gamma) = \min\left[\gamma, \sum_1^n \mathbf{P}(|X_i^s| \geqslant y)\right], \quad \beta_s = \mathbf{P}(|S_n^s| \geqslant \alpha). \tag{33}$$

Further, from Lemma 5 we obtain

$$\mathbf{P}(|X_j^s| \geqslant 2y) \leqslant 2\mathbf{P}(|X_j| \geqslant y). \tag{34}$$

Therefore,

$$P_s(2y, 2\gamma) \leqslant 2P(y, \gamma).$$

From here

$$P_{1s}(y, 2\alpha, 2\gamma) \leqslant \sum_1^{k_{2\alpha}-1} 2^k k P\left(\frac{k_{2\alpha} - k}{k}\alpha, \gamma\right)\gamma^{k-1} + (2\gamma)^{k_{2\alpha}}. \tag{35}$$

A consequence of the inequality (31) is

$$\beta_s \leqslant 2\beta. \tag{36}$$

From (35) and (36) there follows that

$$P_{1s}(y, 2\alpha, 2\beta_s) \leqslant P_5(y/2\alpha, \alpha, 2\beta). \tag{37}$$

Formulas (30), (32), and (36) lead to the estimate (29).

COROLLARY 7. If $\beta < 1/4$, then for any $t > 0$ and $y \geqslant 2\alpha$ we have

$$\mathbf{P}(|S_n| \geqslant y) \leqslant \frac{4}{3} c(t, 2\beta) A_t \Big/ \left(\frac{y-\alpha}{2}\right)^t + \frac{1}{2}(4\beta)^{(y-\alpha)/4\alpha - 2}\left(\frac{y-\alpha}{\alpha}\right)^2, \tag{38}$$

where $c(t, \gamma) = 2^{t-1}\gamma^{-2}(-\ln 2\gamma)^{-t-2}\Gamma(t+2)$.

Proof. First of all, for $y \geqslant 2$ and $2\gamma < 1$ we have

$$P_5(y, \alpha, \gamma) \leqslant \frac{1}{2}\gamma^{-2}\int_1^{y-1}(2\gamma)^u u P\left(\frac{y-u-1}{u}\alpha, \gamma\right)du + (2\gamma - 1 + y)(2\gamma)^{y-2}$$

$$\leqslant \frac{1}{2}\gamma^{-2}\int_0^{y-1}(2\gamma)^u u P((y-u-1)\alpha/u, \gamma)\,du + (2\gamma)^{y-2}y. \tag{39}$$

For u < y/2 − 1, making use of the estimate $P\left(\frac{y-u-1}{u}\alpha, \gamma\right) < (2u/y\alpha)^t A_t$, for γ < 1/2 we obtain

$$\int_0^{y/2-1} \leqslant (2/y\alpha)^t(-\ln 2\gamma)^{-t-2}\Gamma(t+2)A_t. \tag{40}$$

On the other hand, by virtue of the inequality (4), we have

$$(\forall\,(\gamma<1/2))\left(\int_{y/2-1}^{y-1} \leqslant y^2(2\gamma)^{y/2}\right). \tag{41}$$

From (39)-(41) there follows that

$$(\forall\,(y\geqslant 4))\,(P_5(y, \alpha, \gamma)\leqslant c(t, \gamma)(\alpha y)^{-t}A_t+\frac{3}{2}y^2(2\gamma)^{y/2-2}, \quad \gamma<1/2), \tag{42}$$

where $c(t, \gamma)=2^{t-1}\gamma^{-2}(-\ln(2\gamma))^{-t-2}\Gamma(t+2)$. Estimating P_5 in (29) with the aid of (42), we obtain inequality (38) for $y\geqslant 5\alpha$.

For $2\alpha\leqslant y<5\alpha$ the estimate (38) is trivial.

COROLLARY 8. Let β < 1/4 and assume that the function g is the same as in Corollary 2. Then

$$(\forall\,(y\geqslant 2\alpha))\left(\mathbf{P}(|S_n|\geqslant y)\leqslant P_6\left(\frac{y-\alpha}{2\alpha}, \alpha, 4\beta\right)+(4\beta)^{\frac{y-\alpha}{4\alpha}-2}\left(\frac{y-\alpha}{\alpha}\right)^2\Big/2\right), \tag{43}$$

where $P_6(y, \alpha, \gamma)=\gamma^{-2}A_g(y^2/3)\exp\left\{-\max\left[g\left(\frac{1}{u_0\left(\frac{y\alpha}{2}\ln\gamma\right)}\right), g(\alpha)\right]\right\}$, $u_0(a)$ is the root of the equation $g(1/u)+au=0$.

Proof. Without loss of generality, we can assume that $y\geqslant 3\alpha$. Applying inequalities (39)-(41) and

$$P\left(\frac{y-u-1}{u}\alpha, \gamma\right)\leqslant A_g e^{-g(y\alpha/2u)},$$

u < y/2 − 1, we obtain the estimate

$$P_5(y, \alpha, \gamma)\leqslant\frac{1}{2}\gamma^{-2}A_g\int_0^{y/2}u\exp\{-g(\alpha y/2u)+u\ln(2\gamma)\}\,du+\frac{8}{2}y^2(2\gamma)^{y/2-2}, \qquad \gamma<1/2,\ y\geqslant 2. \tag{44}$$

Further,

$$I\equiv\int_0^{y/2}u\exp\{-g(y\alpha/2u)+u\ln(2\gamma)\}\,du<\frac{y^2}{8}\exp\{-g(\alpha y/2u_1)\}, \tag{45}$$

where u_1 is the root of the equation

$$g(y\alpha/2u)+u\ln(2\gamma)=0. \tag{46}$$

We have made use of the inequality

$$g(y\alpha/2u_1)-u_1\ln(2\gamma)\leqslant 2(g(\alpha y/2u)-u\ln(2\gamma)).$$

If $g(\alpha)+\frac{y}{2}\ln(2\gamma)>0$, then

$$I\leqslant\frac{y^2}{8}\exp\{-g(\alpha)\}. \tag{47}$$

By the substitution u = yαv/2, Eq. (46) turns into the equation

$$g(1/v)+(y\alpha v/2)\ln 2\gamma=0.$$

Thus,

$$u_1=\frac{y\alpha}{2}u_0\left(\frac{y\alpha}{2}\ln(2\gamma)\right). \tag{48}$$

From formulas (44)-(48) there follows that

$$P_5(y, \alpha, 2\beta)\leqslant\frac{3}{4}P_6(y, \alpha, 4\beta)+2y^2(4\beta)^{y/2-2}, \quad y\geqslant 4. \tag{49}$$

Combining the estimates (29) and (49), we obtain the inequality (43).

THEOREM 3. Let $\beta < 1/4$. Then for any integer $m > 1$ and $y \geqslant (8m+1)\alpha$ we have

$$\mathbf{P}(|S_n| \geqslant y) \leqslant \frac{4}{3} P_5^m\left(\frac{y-\alpha}{8m\alpha}, \alpha, 2\beta\right) + \frac{8}{3}(1-4\beta)^{-1} \sum_1^n \mathbf{P}(|X_j| \geqslant (y-\alpha)/4m). \tag{50}$$

Proof. Estimating $\mathbf{P}(|S_n^s| \geqslant y-\alpha)$ in the inequality (30) by means of (27) and taking into account (32), for $y \geqslant (8m+1)\alpha$ we have

$$\mathbf{P}(|S_n| \geqslant y) \leqslant \frac{4}{3}\left(P_{1s}^m\left(\frac{y-\alpha}{4m}, 2\alpha, 2\beta_s\right) + (1-2\beta_s)^{-1} \sum_1^n \mathbf{P}\left(|X_j^s| \geqslant \frac{y-\alpha}{2m}\right)\right), \tag{51}$$

where P_{1s} and β_s are defined by the equalities (33). From (51), by virtue of the formulas (34), (36), (37), there follows the assertion of Theorem 3.

COROLLARY 9. If $\beta < 1/4$, then for any $t > 0$, integers $m > 1$ and $y \geqslant (8m+1)\alpha$ we have

$$\mathbf{P}(|S_n| \geqslant y) \leqslant \frac{4}{3}\left(c(t, 2\beta)\left(\frac{y-\alpha}{8m}\right)^{-t} A_t + 2\left(\frac{y-\alpha}{8m\alpha}\right)^2 (4\beta)^{\frac{y-\alpha}{16m\alpha}-2}\right)^m + \frac{8}{3}(1-4\beta)^{-1} \sum_1^n P(|X_j| \geqslant (y-\alpha)/4m), \tag{52}$$

where $c(t, \beta)$ is the same as in the inequality (38).

Proof. As a consequence of (42), we have

$$P_5\left(\frac{y-\alpha}{8m\alpha}, \alpha, 2\beta\right) \leqslant c(t, 2\beta)\left(\frac{8m}{y-\alpha}\right)^t A_t + 2\left(\frac{y-\alpha}{8m\alpha}\right)^2 (4\beta)^{\frac{y-\alpha}{16m\alpha}-2}, \quad y \geqslant (8m+1)\alpha.$$

It remains to apply Theorem 3.

COROLLARY 10. Suppose that the assumptions of Corollary 8 are satisfied. Then for integers $m > 1$ and $y \geqslant (8m+1)\alpha$ we have

$$\mathbf{P}(|S_n| \geqslant y) \leqslant \frac{4}{3}\left(\frac{3}{4} P_6\left(\frac{y-\alpha}{8m\alpha}, \alpha, 4\beta\right) + 2\left(\frac{y-\alpha}{8m\alpha}\right)^2 (4\beta)^{\frac{y-\alpha}{16m\alpha}-2}\right)^m + (8/3)(1-4\beta)^{-1} \sum_1^n \mathbf{P}(|X_j| \geqslant (y-\alpha)/4m).$$

Proof. The required inequality follows from the formulas (49), (50).

THEOREM 4. For any $t \geqslant 1$ we have

$$\mathbf{E}M_n^t \leqslant c_1(t, \delta)(A_t + t\alpha^t),$$

where $c_1(t, \delta) = 2^{t-1}\Gamma(t+2)\delta^{-3}(-\ln\delta)^{-t-2}$.

Proof. Obviously,

$$\mathbf{E}M_n^t = t\int_0^\infty y^{t-1}\mathbf{P}(M_n \geqslant y)\, dy. \tag{53}$$

By virtue of the second of the inequalities (1) we have

$$\int_0^\infty y^{t-1}\mathbf{P}(M_n \geqslant y)\, dy \leqslant \delta^{-2}\int_0^\infty u\delta^u du \int_{(u+1)\alpha}^\infty y^{t-1} P\left(\frac{y-(u+1)\alpha}{u}\right) du + \frac{2}{\alpha}\int_0^\infty y^t \delta^{y/\alpha-2} dy. \tag{54}$$

Further,

$$I \equiv \int_{(u+1)\alpha}^\infty y^{t-1} P\left(\frac{y-(u+1)\alpha}{u}\right) dy = u\int_0^\infty (uz + (u+1)\alpha)^{t-1} P(z)\, dz$$

$$\leqslant 2^{t-2}\left(u^t \int_0^\infty z^{t-1} P(z)\, dz + \alpha^{t-1}(u+1)^t \int_0^\infty P(z)\, dz\right).$$

Since

$$(\forall\, (0 \leqslant f \leqslant 1,\ t > 1,\ \alpha > 0))\left(\int_0^\infty f(u)\, du \leqslant \alpha + \alpha^{1-t}\int_\alpha^\infty f(u)\, u^{t-1} du\right), \tag{55}$$

we obtain the estimate

$$I \leqslant 2^{t-2}((u^t + (u+1)^t)A_t/t + (u+1)^t\alpha^t). \tag{56}$$

From (53), (54), and (56), with the aid of the estimate

$$\int_0^\infty u(u+1)^t\delta^u du < \delta^{-1}\Gamma(t+2)(-\ln\delta)^{-t-2}$$

and the relation

$$\int_0^\infty y^t\delta^{y/\alpha-2}dy = \delta^{-2}\alpha^{t+1}\Gamma(t+1)(-\ln\delta)^{-t-1}$$

we obtain the assertion of the theorem.

THEOREM 5. For any $\beta < 1/4$ and $t > 1$ we have

$$\mathbf{E}|S_n|^t \leqslant c_2(t, \beta)(A_t + (t+1)\alpha^t), \tag{57}$$

where $c_2(t, \beta) = 2^{2t-2}(t+1)c_1(t, 4\beta)/(1-\beta)$ [$c_1(t, \beta)$ is defined in Theorem 4].

Proof. It is easy to see that

$$\mathbf{E}|S_n|^t = t\int_0^\infty y^{t-1}P(|S_n| \geqslant y)\,dy \leqslant \alpha^t + t\int_\alpha^\infty y^{t-1}\mathbf{P}(|S_n| \geqslant y)\,dy \equiv \alpha^t + tI(\alpha).$$

As a consequence of inequality (30) we have

$$I(\alpha) \leqslant \frac{1}{1-\beta}\int_0^\infty \mathbf{P}(|S_n^s| \geqslant y)(y+\alpha)^{t-1}dy.$$

From these relations, taking into account (55) we conclude that

$$\mathbf{E}|S_n|^t \leqslant \frac{2^{t-2}}{1-\beta}\left(\mathbf{E}|S_n^s|^t + \frac{3}{2}\alpha^t\right)(t+1). \tag{58}$$

Obviously,

$$\mathbf{E}|S_n^s|^t \leqslant \mathbf{E}|M_n(s)|^t,$$

where $M_n(s) = \max\limits_{1 \leqslant k \leqslant n} |S_k^s|$. By virtue of the inequalities (23), (28), we have

$$\mathbf{P}(M_n(s) \geqslant \alpha) \leqslant 2\mathbf{P}(|S_n^s| \geqslant \alpha) \leqslant 4\mathbf{P}(2|S_n| \geqslant \alpha) = 4\beta.$$

Applying now Theorem 4 to $\mathbf{E}|M_n(s)|^t$, we obtain

$$\mathbf{E}|S_n^s|^t \leqslant 2^{t-1}c_1(t, 4\beta)(A_t + t\alpha^t). \tag{59}$$

Combining (58) and (59), we obtain inequality (57).

If B has type 2, then we denote by c(B) a constant for which for any n we have

$$\mathbf{E}\left|\sum_1^n Z_j\right|^2 \leqslant c(B)\sum_1^n \mathbf{E}|Z_j|^2,$$

where Z_j are arbitrary independent random variables with values in B with $\mathbf{E}Z_j = 0$.

COROLLARY 11. Assume that B has type 2, $\mathbf{E}X_j = 0$, $j = 1 \div n$, and $1 \leqslant t \leqslant 2$. Then

$$\mathbf{E}|S_n|^t \leqslant 55(t+1)2^t c_2(t, 1/5)c(B)A_t$$

[$c_2(t, \beta)$ is defined in Theorem 5].

Proof. We shall make use of inequality (57). To this end we select α so that $\beta < 1/4$. Let

$$X_k' = X_k \operatorname{Ind}_\alpha(|X_k|), \quad X_k'' = X_k - X_k',$$

where $\operatorname{Ind}_\alpha(\cdot)$ is the indicator of the set $\{y\colon |y| < \alpha,\ y \in R_1\}$.

It is easy to see that

$$\mathbf{P}(|S_n| \geqslant \alpha) \leqslant \mathbf{P}(|S_n'| \geqslant \alpha) + \sum_1^n \mathbf{P}(|X_k| \geqslant \alpha), \tag{60}$$

where $S_n' = \sum_1^n X_k'$. Obviously,

$$\mathbf{P}(|S_n'| \geqslant \alpha) \leqslant \mathbf{E}|S_n'|^2/\alpha^2. \tag{61}$$

Further,

$$E|S_n'|^2\leqslant 2(\mathbf{E}|S_n'-\mathbf{E}S_n'|^2+|\mathbf{E}S_n'|^2). \tag{62}$$

Since $\mathbf{E}S_n'=-\sum_1^n \mathbf{E}X_k''$, we have the estimate

$$|\mathbf{E}S_n'|\leqslant\alpha^{1-t}A_t. \tag{63}$$

On the other hand,

$$\mathbf{E}|S_n'-\mathbf{E}S_n'|^2\leqslant c(B)\sum_1^n \mathbf{E}|X_k'-\mathbf{E}X_k'|^2,$$

$$\mathbf{E}|X_k'-\mathbf{E}X_k'|^2\leqslant 4\mathbf{E}|X_k'|^2\leqslant 4\alpha^{2-t}\mathbf{E}|X_k|^t.$$

Thus,

$$\mathbf{E}|S_n'-\mathbf{E}S_n'|^2\leqslant 4c(B)\alpha^{2-t}A_t. \tag{64}$$

From (61)-(64) there follows that

$$\mathbf{P}(|S_n'|\geqslant\alpha)\leqslant 2(4c(B)\alpha^{-t}A_t+\alpha^{-2t}A_t^2).$$

The last inequality, together with (60), leads to the estimate

$$\mathbf{P}(|S_n|\geqslant\alpha)\leqslant(8c(B)+1)\alpha^{-t}A_t+2\alpha^{-2t}A_t^2.$$

From here we obtain that for

$$\alpha^t=2^t6(8c(B)+1)A_t$$

we have the estimate

$$\beta\equiv\mathbf{P}(|S_n|\geqslant\alpha/2)\leqslant 1/5.$$

Setting now in (57) $\beta = 1/5$, $\alpha^t = 2^t 6(8c(B) + 1)A_t$ and taking into account that $c(B)\geqslant 1$, we obtain the required inequality.

We introduce the notation $B_n^2=\sum_1^n \mathbf{E}|X_j|^2$.

COROLLARY 12. Assume that B has type 2, $\mathbf{E}X_j=0$, $j=1\div n$. Then

$$(\forall(t\geqslant 2))(\mathbf{E}|S_n|^t\leqslant c_2(t,\ 4/25)(A_t+(t+1)(5c^{1/2}(B)B_n)^t)),$$

where $c_2(\cdot,\ \cdot)$ is defined in Theorem 4.

Proof. Let $\alpha = 5B_n c^{1/2}(B)$. Then

$$\beta\equiv\mathbf{P}(2|S_n|\geqslant\alpha)\leqslant 4c(B)B_n^2/\alpha^2=4/25<1/4.$$

It remains to apply inequality (57) with these values α and β.

Assuming that B has type 2, we derive some estimates for the quantile

$$\alpha(\delta)=\inf\{\alpha:\ \mathbf{P}(|S_n|>\alpha)\leqslant\delta\}.$$

We set $G(y)=\sum_1^n \mathbf{P}(|X_k|>y)$,

$$X_k(y)=\begin{cases}X_k, & |X_k|\leqslant y,\\ 0, & |X_k|>y.\end{cases}$$

It is easy to see that

$$(\forall(y>0,\ u>0))(\mathbf{P}(|S_n|\geqslant u)\leqslant\mathbf{P}(|S_n(y)|\geqslant u)+G(y)),$$

where $S_n(y)=\sum_1^n X_k(y)$. Further,

$$\mathbf{E}|S_n(y)-\mathbf{E}S_n(y)|^2\leqslant 4c(B)\sum_1^n \mathbf{E}|X_k(y)|^2.$$

Setting now $D(y)=\sum_1^n \mathbf{E}|X_k(y)|^2$, $y(p)=\inf\{y:\ G(y)\leqslant p\}$, we obtain that

$$\mathbf{P}(|S_n|\geqslant 2(c(B)D(y(p))/p)^{1/2}+|E(y(p))|)\leqslant 2p,$$

where $E(y)=\mathbf{E}S_n(y)$. Thus,

$$\alpha(\delta)\leqslant 2(2c(B)D(y(\delta/2))/\delta)^{1/2}+|E(y(\delta/2))|.$$

If the random variables X_k are symmetric, then

$$\alpha(\delta) \leqslant 2(2c(B)D(y(\delta/2))/\delta)^{1/2}. \tag{65}$$

Now we derive a lower estimate for $\alpha(\delta)$ in the case of symmetric terms. To this end we make use of the inequality

$$\mathbf{P}(|S_n| > y) \geqslant \frac{1}{2}\mathbf{P}\left(\max_k |X_k| > y\right)$$

(see, for example, [5]).

It is easy to see that

$$\mathbf{P}\left(\max_k |X_k| > y\right) \geqslant \frac{1}{2} G(y),$$

if $G(y) \leqslant 1/2$. Consequently, for $p \leqslant 1/2$ we have

$$\mathbf{P}(|S_n| > y(p)) \geqslant p/4.$$

Therefore, for $\delta \leqslant 1/8$ we have

$$\alpha(\delta) \geqslant y(4\delta). \tag{66}$$

The examples given below show that the lower and upper bounds for $\alpha(\delta)$ in the inequalities (65) and (66) can differ strongly and, moreover, both of them are attainable in a definite sense. In addition, we shall assume that $B = R_1$, while X_k are identically distributed and symmetric. We shall use the notation $\gamma(\delta)$ for constants depending on δ. Correspondingly, c denotes a constant that does not depend on δ. In addition, we set $F(u) = \mathbf{P}(|X_1| < u)$.

<u>Example 1.</u> Assume that $|X_1|$ takes the values 1 and x_0 with probabilities $1 - 1/4n$ and $1/4n$, respectively. Then $D(y) = n - 1/4$ for $1 \leqslant y < x_0$. At the same time $y(p) = 1$ for $1/4 \leqslant p \leqslant 1$, $n > 1$. Therefore, the upper bound of the inequality (65) is equal to $2\sqrt{2}(n-1/4)^{1/2}\delta^{-1/2}$ for $1/2 \leqslant \delta \leqslant 1$. On the other hand,

$$\lim_{n\to\infty} \mathbf{P}\left(|S_n| > u\sqrt{2n}, \max_k |X_k| < x_0\right) = 2e^{-1/4}(1 - \Phi(u)).$$

From here

$$\liminf_{n\to\infty} \alpha(\delta)/\sqrt{n} \geqslant \gamma(\delta),$$

i.e., the bound in the inequality (65) is sharp with respect to the order of increase for $1/2 \leqslant \delta \leqslant 1$.

As far as the lower bound of the inequality (66) is concerned, it turns out that it is understated for $1/16 \leqslant \delta \leqslant 1/4$.

<u>Example 2.</u> Assume that $F(x)$ in the interval $|x| < x_0$ is defined by the density

$$p(x) = c/(1+|x|)^{\nu+1}, \quad 0 < \nu < 2,$$

and for $|x| > x_0$ it is a constant, with the exception of the points $\pm x_1$, $x_1 > x_0$, where it has jumps of magnitude 1/8.

It is easy to see that

$$\limsup_{n\to\infty} \alpha(1/2)/n^{1/\nu} < \infty,$$

if $x_0/n^{1/\nu} \to \infty$.

On the other hand, $y(1/4) = x_0$ and $D(x_0) > cnx_0^{2-\nu}$. Hence for $x_0 = n^{1/\nu+\varepsilon}$ $D^{1/2}(x_0) > cn^{1/\nu+\eta}$, $\eta = (\varepsilon/2)(2-\nu)$, i.e., the upper bound of the inequality (65) is overstated for $\delta = 1/2$.

<u>Example 3.</u> Assume that $F(x)$ is defined by the density

$$p(x) = c/(1+|x|)^{\nu+1}, \quad 0 < \nu < 2.$$

Then for $n \to \infty$ we have

$$y(\delta) \sim \gamma(\delta)n^{1/\nu}, \quad D(y(\delta)) \sim \gamma(\delta)n^{2/\nu},$$

i.e., the lower and the upper bounds have the same order $n^{1/\nu}$.

<u>Comments.</u> The above obtained estimates for $\mathbf{P}(M_n \geqslant y)$ and $\mathbf{P}(|S_n| \geqslant y)$ are new even in the one-dimensional case.

In all the inequalities (as, incidentally, also in [1]) the quantity

$$B_n^2(y)=\sum_1^n \mathbf{E}(\min(y,\ |X_k|))^2$$

is missing and it is replaced by the quantile α.

We note that the ratio $\alpha/B_n(y)$ may be arbitrarily small (see Example 2). In this case, the normal term $\exp\{-c(u|B_n(y))^2\}$, which occurs in many inequalities in [6], becomes considerably larger than $\beta^{u/\alpha}$ if $u<B_n(y)(B_n(y)/\alpha)^{1/2}$, while the ratio u/α is large.

The estimate (2) is sharper than the corresponding inequality from [1] (Theorem 1) because of the dependence on y; on the other hand, in the latter the coefficient of A_t/y^t increases slower when $t\to\infty$.

Corollary 9 is the infinite-dimensional analogue of the inequality (47) from [6]. Other generalization variants of this inequality to the infinite-dimensional case can be found in [7, 8].

Obviously,

$$\exp\{-(u/B_n)^2\}>(A_t/u^t)^\gamma\Rightarrow(u/B_n)^t\exp\{-u^2/\gamma B_n^2\}>A_t/B_n^t.$$

This means that the normal term in the inequalities (46)-(48) of [6] can be dominating for u, large in comparison with B_n only when the Lyapunov ratio A_t/B_n^t is sufficiently small. This agrees well with the central limit theorem.

Since the inequalities (29) and (50) do not contain terms of the form $(A_t/u^t)^\gamma$, they may turn out to be more accurate than the similar inequalities from [6].

We note that the exponential estimate of the form $e^{-\gamma u}$ is attained, for example, when X_k are real, identically distributed, and X_1 takes the values 0 and 1 with the probabilities $1-1/n$ and $1/n$, respectively.

Corollary 10 can be considered as a generalization of the one-dimensional inequality from [9] (see also [10]). First time this inequality has been carried over to the infinite-dimensional case with the preservation of the form in [11]. A more general and at the same time a more accurate form (including the one-dimensional case) has been found in [12].

Theorem 5 differs from Theorem 4 of [1] only by constants.

Corollary 11, within the accuracy of a constant, is a generalization of the von Bahr–Esseen inequality [13], while Corollary 12 can be easily extracted from Theorem 2 in [8] (again, within the accuracy of constants).

Similar inequalities for $\mathbf{E}||S_n|-\mathbf{E}|S_n||^t$, without constraints on $\mathbf{B}$, can be found in [7, 14, 15].

The method applied here goes back to [15, p. 454 of the Russian edition]. An alternate approach has its origin in [16]. It is based on the expansion

$$|S_n|-\mathbf{E}|S_n|=\sum_1^n Y_k, \tag{67}$$

where

$$Y_k=\mathbf{E}\{|S_n|/\mathcal{F}_k\}-\mathbf{E}\{|S_n|/\mathcal{F}_{k-1}\},\quad 1\leqslant k\leqslant n,$$
$$\mathbf{E}\{|S_n|/\mathcal{F}_0\}=\mathbf{E}|S_n|,$$

and

$$(\forall(t>0))(\mathbf{E}\{|Y_k|^t/\mathcal{F}_{k-1}\}\leqslant 2^t\mathbf{E}|X_k|^t). \tag{68}$$

For t = 2 we have a more accurate estimate

$$\mathbf{E}\{|Y_k|^2/\mathcal{F}_{k-1}\}\leqslant\inf_{x\in B}\mathbf{E}|X_k+x|^2 \tag{69}$$

(see [17]).

We note that the random variables X_k form a martingale-difference. Therefore, the expansion (67) in combination with the estimates (68) and (69) allows us to use probability inequalities for martingales (regarding the latter, see, for example, [18, 19]). This circumstance has been mentioned for the first time in [7] and, almost simultaneously, in [11].

Later, similar considerations have been used in [14]. At the described approach it is more natural and convenient to derive inequalities for $\mathbf{P}(||S_n|-\mathbf{E}|S_n||>y)$ and $\mathbf{E}||S_n|-\mathbf{E}|S_n||^t$, than for $\mathbf{P}(|S_n|>y)$ and $\mathbf{E}|S_n|^t$. This is done in the already mentioned investigations [7, 14].

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